Experimental Design for Minimizing Parameter Uncertainty in the Physical and Engineering Sciences

Motivation

The physical and engineering sciences differ from much of life and social sciences in that the equations commonly used in the former often come from theoretical analyses of the system being observed. By contrast, life and social science relationships commonly are so complex that they frequently are modeled by statistically linear models, usually polynomials, with little theoretical motivation for the form of the equation. The theoretically derived expressions frequently involve parameters in nonlinear ways. This discussion pertains to both linear and nonlinear equations, but it results in a relatively simple and general design for linear systems and a potentially more complex design for nonlinear systems. The advantage of such a design is illustrated in the following example, which uses the function $y = \beta[1 - \exp(-\gamma x)]$ as a fitting function (Figure 1), which describes species formation through radioactive decay or other first-order reactions among many other things.

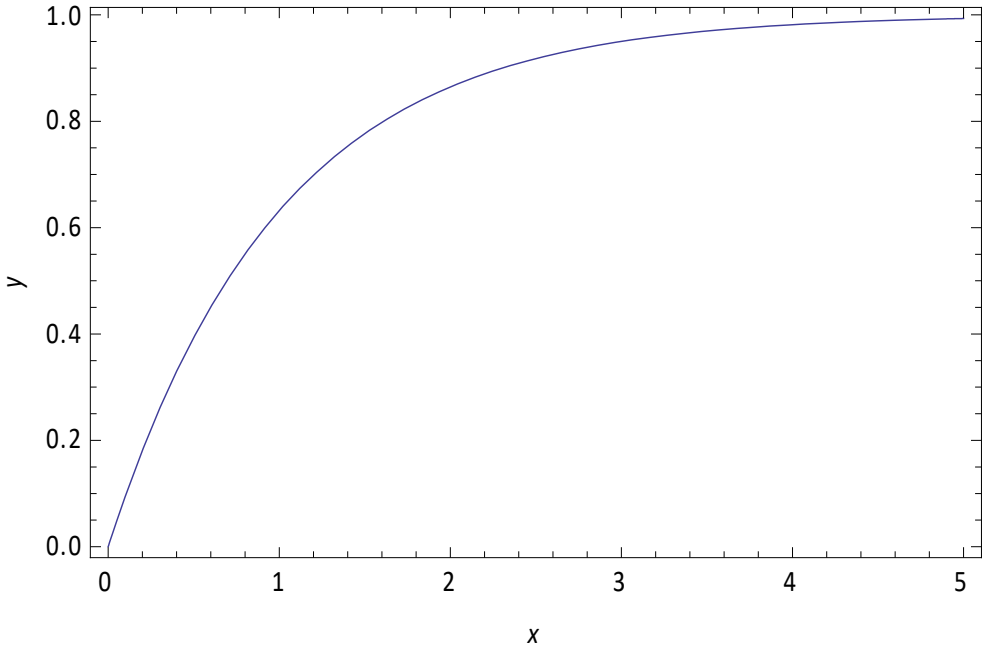
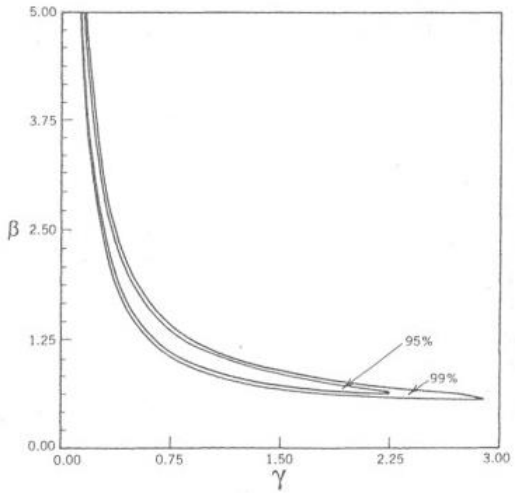


Figure 1 Plot of $y = \beta[1 - \exp(-\gamma x)]$ over the range of significant variation in its parameters.

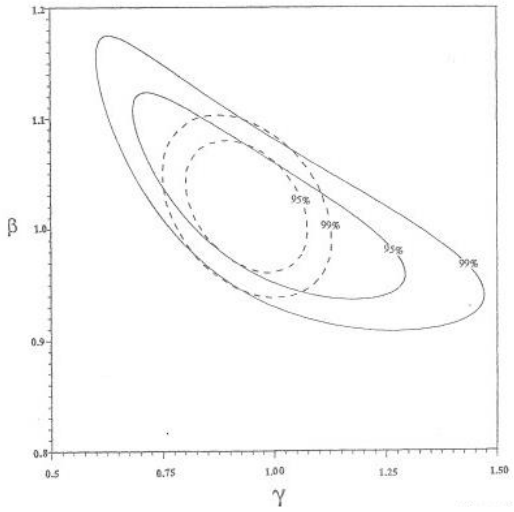
The advantages of rigorously designed experiments compared to casual design appear graphically in Figure 2 in terms of parameter confidence regions. In all cases illustrated, ten experimental data points are used to estimate the equation's parameters with equal error in all measurements and the true value of both parameters is 1. In the first case, the data points are evenly spaced in the region where y is

changing the fastest (x varying from 0 to 1), resulting in the indicated parameter estimates and confidence intervals. If the experimental x values from 1 to 4.6, the confidence intervals shrink considerably, as indicated in the bottom left and top right. If five replicated data at each of two optimally chosen locations (x appx. equal to 1 and 4.6) are used, the confidence intervals shrink further.

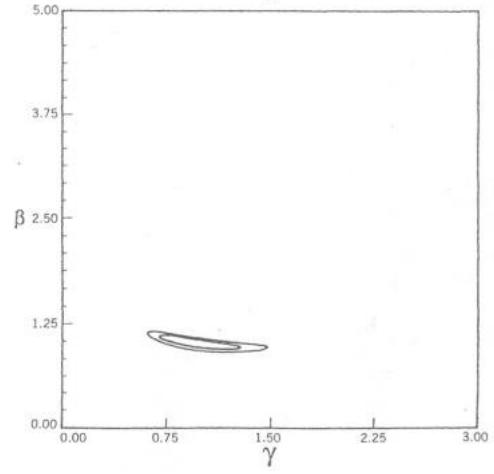
Uniformly spaced experimental data collected in region of most rapid change in results



Uniformly spaced experimental data over broader range (magnified)



Uniformly spaced experimental data over broader range



Optimally spaced experimental data (replicated points near upper and lower limits)

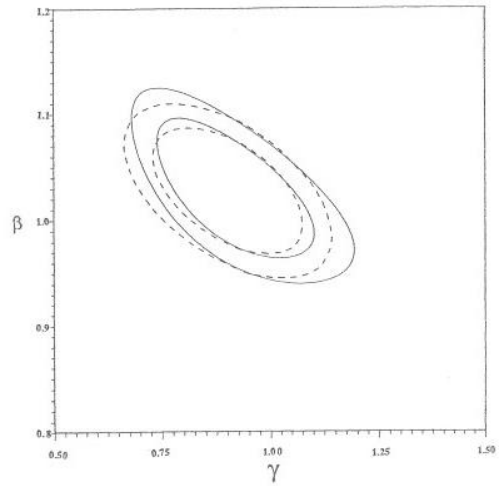


Figure 2 Confidence intervals (95% and 99%) for parameters as estimated from three experimental designs, all with same number of data points and same error but with data from differing values of independent variable in the equation $y = \beta[1 - \exp(-\gamma x)]$ as follows: equally spaced points over the region of most rapid change (top left), replicated points at two values of independent variable (top right), same as top right but at expanded scale (bottom left), at two optimized points (bottom right).

The experimental design that leads to the least uncertainty in parameter values for a given number of measurements and a fixed amount of error in the measurements can be found by the following general procedure.

1. The data will be collected at a number of values of the independent variable as there are parameters and that additional data points will be replicated at these values. That is, if there are two parameters, all of the data will be collected at one of two values of the independent variable.
2. Form an F-matrix from the model equation by taking its derivative with respect to the parameters and evaluating it at each value of the independent variable. In the context of vector calculus generally, this matrix is commonly called the Jacobian matrix.
3. Find the determinant (sometimes called the Jacobian) of this matrix.
4. Determine the values of the independent variable that will maximize the absolute value of this determinant. This is equivalent to maximizing the square of the determinant, but it is commonly easier to maximize its absolute value.

The following illustrations may be helpful.

Statistically Linear Systems (Illustrated by a Second-order Polynomial – three parameters)

A second-order polynomial is a statistically linear model, meaning that the derivatives of the model with respect to its parameters are all independent of the values of the parameters. Assuming the independent variable is temperature, designated by T, we have

$$y = a + bT + cT^2$$

This model has three parameters, a, b and c. Therefore, the optimal design of the experiment will involve measurements at three values of T. The F. matrix for this model, assuming three data points, is

$$\begin{bmatrix} \left.\frac{\partial y}{\partial a}\right|_{T_1} & \left.\frac{\partial y}{\partial b}\right|_{T_1} & \left.\frac{\partial y}{\partial c}\right|_{T_1} \\ \left.\frac{\partial y}{\partial a}\right|_{T_2} & \left.\frac{\partial y}{\partial b}\right|_{T_2} & \left.\frac{\partial y}{\partial c}\right|_{T_2} \\ \left.\frac{\partial y}{\partial a}\right|_{T_3} & \left.\frac{\partial y}{\partial b}\right|_{T_3} & \left.\frac{\partial y}{\partial c}\right|_{T_3} \end{bmatrix} = \begin{bmatrix} 1 & T_1 & T_1^2 \\ 1 & T_2 & T_2^2 \\ 1 & T_3 & T_3^2 \end{bmatrix}$$

The determinant of which is

$$-(T_1 - T_2)(T_1 - T_3)(T_2 - T_3)$$

We can assume, without loss of generality, that T_1 and T_3 represent the lowest and highest achievable temperatures, respectively. These represent the lowest and highest temperatures at which data can be collected without compromising its accuracy, changing the system response so the model is not longer valid, or expending excessive resources. This function (as all similar polynomial functions) increases or decreases without limit as the values of these temperatures change (with the other two remaining

constant). Therefore, the lowest and highest data points should be at the lowest and highest achievable temperatures. The remaining data point should be located where

$$-\frac{d(T_1-T_2)(T_1-T_3)(T_2-T_3)}{d\,T_2}=0=(-T_1+T_2)(T_1-T_3)+(T_1-T_3)(T_2-T_3)$$

and

$$-\frac{d^2(T_1-T_2)(T_1-T_3)(T_2-T_3)}{d\,T^2}<0$$

This is at the point

$$T_2=\frac{1}{2}(T_1+T_3)$$

where the second derivative is

$$2(T_1-T_3)$$

which would always be negative under the assumption that $T_3 > T_1$ and hence always be a maximum. Since we are looking for a maximum in the absolute value, it is not equally valid show that it is always a minimum. The most important thing is that the value can never be zero, which would correspond to an inflection point and would produce a minimum rather than a maximum in absolute value.

Therefore, data should be evenly distributed over these three points to minimize the uncertainty in the parameters. Note that, for this linear function in a statistical sense, the magnitude of the determinant and hence of the points to make the measurements does not depend on the value of the parameters, as would be expected. This result is also general in the sense that statistically linear polynomial functions generally lead to optimal experimental designs in which the data are collected at equally spaced data points, one point per parameter in the polynomial, with the upper and lower limits and the experimental upper and lower limits of the system.

Nonlinear systems are very different, as indicated below.

Arrhenius Reactivity Expression (two parameters)

$$k=k^0\exp\left(-\frac{E}{RT}\right)$$

The F. matrix for this problem, assuming two data points, is

$$\begin{bmatrix}\left.\frac{\partial k}{\partial k^0}\right|_{T_1} & \left.\frac{\partial k}{\partial E}\right|_{T_1}\\ \left.\frac{\partial k}{\partial k^0}\right|_{T_2} & \left.\frac{\partial k}{\partial E}\right|_{T_2}\end{bmatrix}$$

yielding the following matrix

$$\begin{bmatrix} \exp\left(-\frac{E}{RT_1}\right) & -\frac{k^0\ \exp\left(-\frac{E}{RT_1}\right)}{T_1} \\ \exp\left(-\frac{E}{RT_2}\right) & -\frac{k^0\ \exp\left(-\frac{E}{RT_2}\right)}{T_2} \end{bmatrix}$$

the determinant of which is

$$\frac{k^0 \exp\left[-\frac{E}{R}\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\right](T_2 - T_1)}{T_1 T_2}$$

Since the absolute value of this function increases without limit with increasing T_1 or T_2, the maximum of the absolute value will occur when one temperature, say T_2, is at its upper bound, designated T_b from this point. The other maximum occurs where the derivative with respect to T_1 is zero, namely,

$$-\frac{\exp\left[-\frac{E\left(\frac{1}{T_1}+\frac{1}{T_2}\right)}{R}\right]}{T_1^2} = 0$$

or

$$T_1 = \frac{ET_2}{E + RT_2} = \frac{ET_b}{E + RT_\mathrm{b}}$$

The reaction rate coefficient takes on a value of

$$k(T_2) = k^0 \exp\left(-\frac{E}{R\frac{E\ T_b}{E + R\ T_b}}\right) = k^0 \exp\left(-\frac{E + R\ T_b}{R\ T_b}\right) = k^0 \exp\left(-\frac{E}{R\ T_b} - 1\right) = \frac{k(T_\mathrm{b})}{e}$$

That is, the temperature at which the reaction rate coefficient decreases by a factor of $\frac{1}{e}$ from its value at the upper limit is the optimal temperature for the second set of data to be collected.

Note that the optimal design depends on the values of the parameters for this nonlinear expression, as is generally the case for nonlinear systems. Since the objective of the experimental design is generally to determine these values, this is a little frustrating. However, if the values of the parameters are approximately known, or if some preliminary experimentation can lead to reasonable approximates, the design provides useful guidance in selecting the values of the independent variables.

For a typical reaction (34 kcal/mol activation energy) with an upper temperature limit of 450 K, the optimal temperature for the second temperature is 438.5 K – surprisingly close to the upper limit in my mind. As the upper temperature limit or activation energy increases, the lower temperature limit also increases.

A plot of how pronounced the maximum is for typical temperatures and activation energies is below (specific data are for 34 kcal/mol activation energy and an upper temperature bound of 450 K).

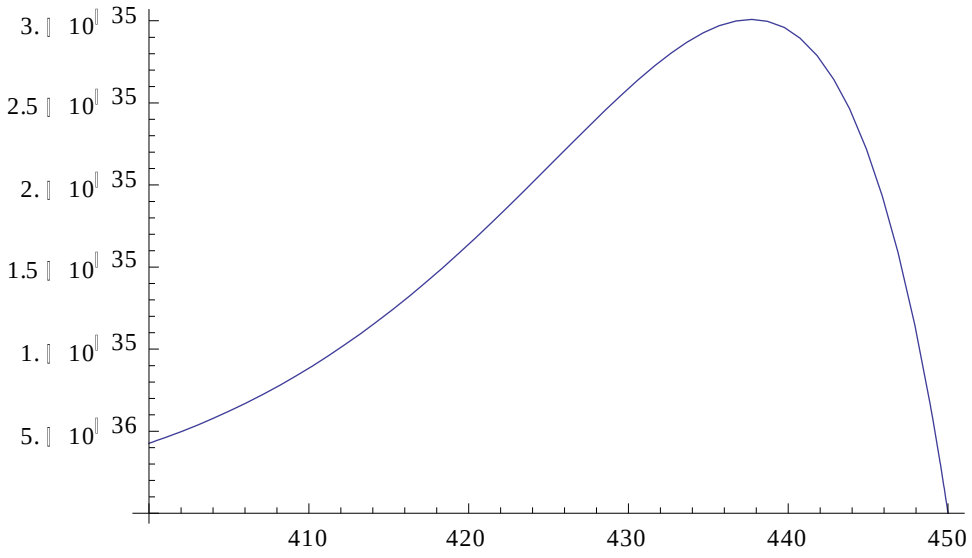


Figure 3 Curve shape of the determinant assuming one experimental value is fixed at its experimental upper bound of 450.

The curve maximizes about 2 % below the point it drops to zero (438.5 K in this case) and drops rapidly as temperature increases and more slowly as temperature decreases. Intuition suggests this trend is correct, since one cannot determine parameters if the two temperatures are equal, but the proximity of the maximum in the curve to the peak temperature is not intuitive (to this author). Since the precise activation energy is not generally known at the point the experiment is being designed, it may be prudent to be slightly conservative in estimating the optimal temperature (choose a point slightly lower than the computed optimal point).

Power Law Correlation

A power law expression such as

$$y = ax^b$$

provides a correlation for many expressions. The F-matrix for this experimental design is as follows

$$\begin{bmatrix} \left.\frac{\partial y}{\partial \mathrm{a}}\right|_{x_1} & \left.\frac{\partial y}{\partial \mathrm{a}}\right|_{x_2} \\ \left.\frac{\partial y}{\partial b}\right|_{x_1} & \left.\frac{\partial y}{\partial b}\right|_{x_2} \end{bmatrix} = \begin{bmatrix} x_1^b & x_2^b \\ ax_1^b \ln x_1 & ax_2^b \ln x_2 \end{bmatrix}$$

with a resulting determinant of

$$ax_1^b x_2^b (\ln x_2 - \ln x_1)$$

If we assume x_2 is larger than x_1, which does not reduce the generality of the approach, the above function clearly increases without bound with increasing x_2 if b is not negative. For negative b, the

function tends to zero with increasing x_2 while it decreases without bound in decreasing x_1. Therefore, one of the two measurement points should be at the highest achievable value of x if b is positive or at the lowest achievable value of x if b is negative. The second value is where the derivative of the above function is zero, which is given by

$$x_1 = x_b \exp\left(-\frac{1}{b}\right) \quad b > 0$$
$$x_2 = x_b \exp\left(-\frac{1}{b}\right) \quad b < 0$$

where x_b is the value bound by the experimental conditions as discussed above.

Lorentzian Function

The preceding two examples were only nonlinear in one parameter even though they involved two parameters. The Lorentzian function describes many physical and stochastic processes and is nonlinear in both of its two parameters. It appears in many forms, but here we will use the simple form below

$$y = \frac{1}{1 + \frac{(x-a)^2}{b^2}}$$

This function describes a bell-shaped curve centered at a and with a characteristic width of b (Figure 4). This function is the solution to a partial differential equation describing constrained resonance, as in, for example, homogeneously broadened line shapes of molecular emission and absorption spectra or resonances in nuclear cross sections and other profiles. The line shape does not account for relativistic effects in these applications. This function is also probability distribution function if normalized by ab in which context it is generally called a Cauchy or a Cauchy-Lorentz distribution. It is also a commonly used empirical function for systems that have characteristic peaks with monotonically declining values at either side of the peak.

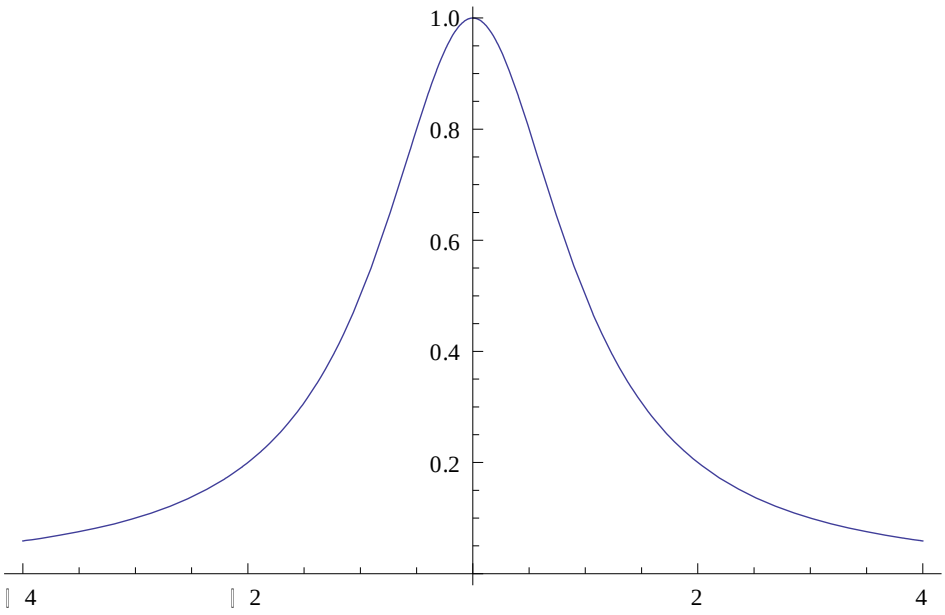


Figure 4 Shape of a Lorentzian function with $a = 0$ and $b = 1$.

The F-matrix for this expression is

$$\begin{bmatrix} \left.\frac{\partial y}{\partial \mathrm{a}}\right|_{x_1} & \left.\frac{\partial y}{\partial \mathrm{a}}\right|_{x_2} \\ \left.\frac{\partial y}{\partial b}\right|_{x_1} & \left.\frac{\partial y}{\partial b}\right|_{x_2} \end{bmatrix} = \begin{bmatrix} \frac{2(-a+x_1)}{b^2(1+\frac{(-a+x_1)^2}{b^2})^2} & \frac{2(-a+x_1)^2}{b^3(1+\frac{(-a+x_1)^2}{b^2})^2} \\ \frac{2(-a+x_2)}{b^2(1+\frac{(-a+x_2)^2}{b^2})^2} & \frac{2(-a+x_2)^2}{b^3(1+\frac{(-a+x_2)^2}{b^2})^2} \end{bmatrix}$$

The determinant of this matrix is

$$-\frac{4b^3(a-x_1)(a-x_2)(x_1-x_2)}{(b^2+(a-x_1)^2)^2(b^2+(a-x_2)^2)^2}$$

Unlike the previous examples, this function does not increase or decrease without bound as the values of x_1 and x_2 increase or decrease and, in fact, the function tends to zero for all finite values of the parameters a and b as x_1 and x_2 tend to $\pm\infty$, as one would expect since the denominator involves higher powers of x_1 and x_2 than the numerator. Therefore, unlike the previous examples, the optimum values of x_1 and x_2 at which to collect data will be bounded by theory. If these bounds are experimentally achievable, there will be no experimentally dictated upper or lower limits as was the case with the earlier examples.

The derivatives of this determinant with respect to the values x_1 and x_2 are

$$-\frac{4b^3(a-x_2)\{a^3+a[b^2-3x_1(x_1-2x_2)]+x_1^2(2x_1-3x_2)-3a^2x_2+b^2(-2x_1+x_2)\}}{[b^2+(a-x_1)^2]^3[b^2+(a-x_2)^2]^2}$$

and

$$\frac{4b^3(a-x_1)[a^3+ab^2-3a^2x_1+b^2x_1-2(b^2-3ax_1)x_2-3(a+x_1)x_2^2+2x_2^3]}{[b^2+(a-x_1)^2]^2[b^2+(a-x_2)^2]^3}$$

respectively.

Setting these to zero is and solving for x_1 and x_2 yields a total of nine solutions or roots since x_1 and x_2 appear in the expressions at powers up to nine. Three of these are repeated roots where x_1 and x_2 both equal a, which represents a minimum in the absolute value of the determinant rather than a maximum. Two of the maxima are at

$$x_1 = a \pm \sqrt{\frac{3}{5}}b, x_2 = a \mp \sqrt{\frac{3}{5}}b$$

and the remaining four are at

$$x_1 = a \pm \frac{1}{2}\sqrt{\frac{7+\sqrt{33}}{2}}\sqrt{b^2},\ \ x_2 = a \pm \frac{1}{2}\sqrt{\frac{7-\sqrt{33}}{2}}\sqrt{b^2}$$

and

$$x_1 = a \pm \frac{6\left(3\sqrt{3} + 7\sqrt{11}\right)}{\left(15 + \sqrt{33}\right)^2}\sqrt{b^2}, \quad x_2 = a \pm \frac{1}{2}\sqrt{\frac{7 + \sqrt{33}}{2}}\sqrt{b^2}$$

Since we do not know if b is positive or negative, we cannot write $\sqrt{b^2} = b$ ($\sqrt{b^2} = -b$ if b is negative). The second form is approximately

$$x_1 = a + 0.396143\sqrt{b^2}, x_2 = a + 1.26217\sqrt{b^2}$$

Contour and surface plots of the determinant are often the best ways to distinguish global from local maxima. The plots appear below assuming a = 0 and b = 1. Characteristic of all determinants of this type, the plots are symmetric about the line $x_1 = x_2$, which reflects the idea that x_1 and x_2 are interchangeable, and it is arbitrary which is considered larger/smaller. For this reason, it does not matter how the axes are labeled, and they are unlabeled here. Additionally, the value of the determinant is zero for $x_1 = x_2$, again characteristic of all determinants of this type, indicating that you can gain no information about either parameter if data are not available at two unique values of the independent variable.

In terms of properties unique to this function and not characteristic of all determinants of this type, the six local maxima and the minimum at the triply repeated root at x_1 and x_2 equal to a are all evident in the plot, although the height of the peak of the highest maxima has been truncated to make the other peaks more visible. Clearly, the two peaks at $x_1 = a \pm \sqrt{\frac{3}{5}}b, x_2 = a \mp \sqrt{\frac{3}{5}}b$ are much greater maxima than the other four and are the optimal choices for this experimental design. Although these are two peaks, they represent a single set of independent variable values, namely, values offset equidistant from the value of a by a distance $\sqrt{\frac{3}{5}}b$.

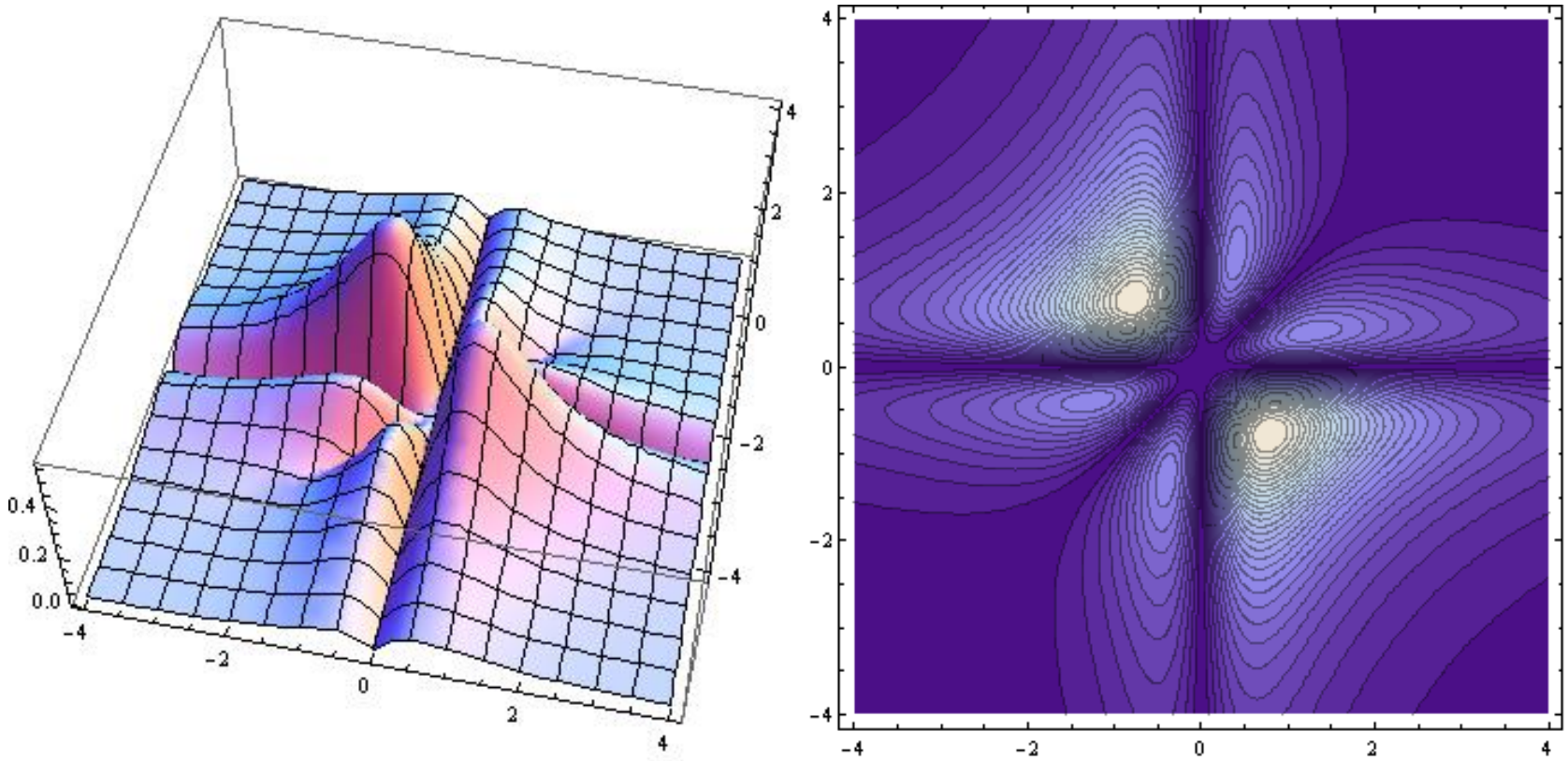


Figure 5 Surface (left) and contour (right) plots of the determinant of the Lorentzian function with $a = 0$ and $b = 1$

A superficial experimental design for this system based on an inspection of Figure 4 might suggest that the peak of the curve is a good choice for one of the values of the independent variables, with the second located perhaps half way up the peak. An examination of the surface/contour plots reveals that the location $\mathrm{x} = a$ is an especially poor choice for conducting experiments since the value of the determinant is near zero at this point for all values of the second point. This highlights one of the advantages of careful experimental design. Unlike many of the previous examples, the peaks in the these curves are relatively narrow, indicating that the uncertainty in the parameters increases rapidly as the experimental data points shift from the local optimum, especially if the shift is toward the value a. If the true value of a is unknown, as would be almost always the case, the curve suggests the best design is to perturb one of the experimental points away from a. Note that perturbing them both equal distances from the estimated values of a is less advantageous than perturbing only one of them if the estimated value of a is equal to or larger than the actual value.

The relative values of the local maxima depend on the values of the parameters chosen, and it is conceivable that the other roots could become the global maxima. This does not happen in this particular case. Changing the value of a shifts the center of the symmetric plot along the $x_1 = x_2$ symmetry line but otherwise does not change it. Changing the value of b stretches the plot along uniformly perpendicular to the $x_1 = x_2$ symmetry line, but does not change the identification of the global maxima.

First-order Generation/Decay Kinetics

Consider the equation used in the introduction of this section, $y = \beta[1 - \exp(-\gamma x)]$, which also has a single nonlinear parameter. Applying the same techniques to this function leads to the following matrix,

$$\begin{bmatrix} \left.\frac{\partial y}{\partial \beta}\right|_{x_1} & \left.\frac{\partial y}{\partial \beta}\right|_{x_2} \\ \left.\frac{\partial y}{\partial \gamma}\right|_{x_1} & \left.\frac{\partial y}{\partial \gamma}\right|_{x_2} \end{bmatrix} = \begin{bmatrix} 1 - e^{-\gamma x_1} & e^{-\gamma x_1}\beta x_1 \\ 1 - e^{-\gamma x_2} & e^{-\gamma x_2}\beta x_2 \end{bmatrix}$$

with a determinant of

$$-e^{-\gamma(x_1+x_2)}\beta((-1 + e^{\gamma x_2})x_1 - (-1 + e^{\gamma x_1})x_2)$$

and derivatives with respect to x_1 and x_2 of

$$e^{-\gamma(x_1+x_2)}\beta(1 - \gamma x_1 + e^{\gamma x_2}(-1 + \gamma x_1) + \gamma x_2)$$

and

$$-e^{-\gamma(x_1+x_2)}\beta(\gamma x_1 + (-1 + e^{\gamma x_1})(-1 + \gamma x_2))$$

respectively.

If we assume $x_1 < x_2$, we can tell by inspection that the determinate absolute value increases without bound as x_2 increases to infinity. Therefore, x_2 should be as large as possible. Solving either of the equations above for x_1 subject to the constraint that $x_1 < x_2$ yields

$$x_1 = \frac{1}{\gamma} - \frac{x_2}{-1 + e^{\gamma x_2}}$$

We could also solve this problem graphically. The methods illustrated for the Lorentzian function are useful here as well. If one plots (or simply evaluates) the determinant at the estimated parameter values, the peaks in the absolute value of the determinant are commonly obvious and, while not available in analytical form, typically can be discerned with sufficient accuracy for a design. In this case, one variable should be as large as possible (though larger than about 5 seems to add little additional merit) and the other value should be about 1.

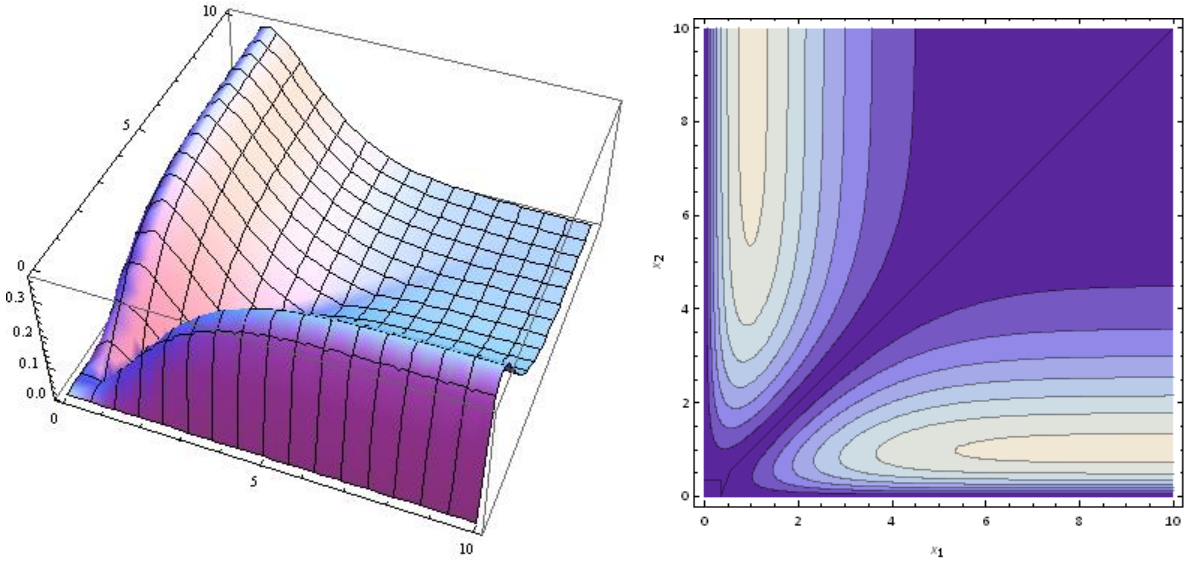


Figure 6 Surface and contour plots of the determinant of the first-order species generation function assuming $\beta = \gamma = 1$.